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On a singular Sturm–Liouville problem in the theory of molecular vibrations

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In this paper, we study the angular part of the internal amplitude function of a diatomic molecule under an angular potential function depending only on cos θ and which is an entire even function on **R**. For the equation

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\psi}{\mathrm{d}\theta}\right)+\left(\lambda-V(\cos\theta)-\frac{m^2}{\sin^2\theta}\right)\psi=0,$$

we obtain a formula for the eigenvalues and we determine the corresponding eigenfunctions.

0. Introduction

We consider a diatomic molecule in the gas state. The time-independent Schrödinger equation of a two-particle system is $\sum_{i=1}^{2}\frac{\hbar^2}{2m_i}\nabla_i^2\psi+(E-V)\psi=0$ where E (resp. V) is the total (resp. potential) energy of the system. Separating the motion of the center of mass under the assumption that the total energy E is the sum of that of the system relative motion and that of the center of mass translational motion and likewise the potential energy of the system is the sum of the external potential acting on the system concentrated at its center of mass and the internal potential which controls the interaction of the two particles we arrive at the equation $\frac{\hbar^2}{2\mu}\nabla^2\psi+(E-V)\psi=0$ where now, by abuse of notation, E (resp. V) is the total (resp. potential) energy of the system relative motion and ψ, again by abuse of notation, is the internal amplitude function and $\mu=\frac{m_1m_2}{m_1+m_2}$ is the reduced mass of the system.

We express the last equation in spherical coordinates r, θ, and we assume the potential energy to be the sum of a central potential depending only on r and an angular potential depending only on cos θ. By separation of variables we concentrate on the angular dependence of the internal amplitude function which takes the form [5, chapter 5]

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}\psi}{\mathrm{d}\theta}\right)+\left(\lambda-V(\cos\theta)-\frac{m^2}{\sin^2\theta}\right)\psi=0,$$

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where m is the magnetic quantum number and ψ (resp. $V(\cos\theta)$) is now, again by abuse of notation, the θ-dependent amplitude function (resp. potential function). Substituting $z = \cos\theta$, we arrive at the equation

$$\frac{\mathrm{d}}{\mathrm{d}z}((1-z^2)\psi') + \left(\lambda - V(z) - \frac{m^2}{1-z^2}\right)\psi = 0.$$

We will give a detailed mathematical analysis of this equation when V is an entire even function in $\mathbf{R}$ and we determine the eigenvalues and the corresponding eigenfunctions. Note that this equation with a quadratic V is also obtained on analyzing the Schrödinger equation of an electron in the field of two protons at rest (the ion H_2^+) [4, p. 279].

In mathematical terms our problem is to study the following Singular Sturm–Liouville problem:

$$\text{Determine all } \lambda \in \mathbf{C} \text{ such that } w'' + \frac{p(z)}{1-z^2}w' + \frac{q(z)}{(1-z^2)^2}w = 0, \tag{0.1}$$

where $p(z) = -2z$, $q(z) = -k - (V(z) - \lambda)(1 - z^2)$ with $k \geqslant 0$ and $V(z)$ real -valued entire even function in $\mathbf{R}$, admits a non- trivial bounded complex-valued solution ϕ_λ in $-1 < z < 1$. These λ's are called the eigenvalues of the Sturm–Liouville problem and the ϕ_λ's are the corresponding eigenfunctions. Note that $V(z) = \sum_{n\geqslant 0} \alpha_{2n} z^{2n}$ entire power series where α_{2n} are real for $n \geqslant 0$.

The paper is divided into two sections. In section 1, we obtain a characterization of bounded solutions in terms of the corresponding eigenvalue in proposition 1.2 and theorem 1.5. In section 2, the work in section 1 will enable us to give a formula for the eigenvalues and the final solution of our problem is given in theorem 2.7. There is an appendix A that contains some detailed computations that are needed for our final solution so as not to interrupt the presentation in the two sections.

1. Characterization of bounded solutions

We study equation (0.1) in the complex plane. Set $\underline{w} = \begin{pmatrix} w \\ w' \end{pmatrix}$ and transform equation (0.1) to the vector form

$$\underline{w}' = \begin{pmatrix} 0 & 1 \\ \frac{-q(z)}{(1-z^2)^2} & \frac{-p(z)}{1-z^2} \end{pmatrix} \underline{w}. \tag{1.1}$$

According to the basic theory of linear differential equations the solutions of equation (1.1) in the simply connected domain $\mathrm{H} = \mathbf{C} - \{x \in \mathbf{R}\colon |x| \geqslant 1\}$ form a two-dimensional vector spaces over $\mathbf{C}$ and $\{\pm 1\}$ are regular singular points for equation (1.1). We call any two linearly independent solutions $\underline{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and

$\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ of equation (1.1) fundamental solutions of equation (1.1) and $\begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$ a fundamental matrix of equation (1.1) and, by abuse of notation, we also call u_1, v_1 fundamental solutions of equation (1.1).

We study the solutions near -1. A similar argument holds near $+1$. Set $\underline{w}_1 = \begin{pmatrix} w \\ (1+z)w' \end{pmatrix}$ so that

$$\underline{w}_1 = \frac{1}{1+z}\begin{pmatrix} 0 & 1 \\ \frac{-q(z)}{(1-z)^2} & 1 - \frac{p(z)}{1-z} \end{pmatrix}\underline{w}_1 = A(z)\underline{w}_1 \tag{1.2}$$

and $A(z)$ has a simple pole at -1. An application of Floquet theorem shows that a fundamental matrix $M(z)$ for (1.2) in $\{z \in \mathbf{H} : |z+1| < 2\}$ is given by [1, p. 392] $M(z) = S(z)\, e^{B \log(1+z)}$ where $S(z)$ is the restriction of an invertible analytic matrix in $\{z \in \mathbf{C} : 0 < |z+1| < 2\}$ that has at most a pole at -1 and $\mathbf{B} = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ or $\begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ so that equation (0.1) has at least one solution in $\{z \in \mathbf{H} : |z+1| < 2\}$ of the form $(z+1)^{\rho} v(z)$ where v is analytic in $\{z \in \mathbf{C} : |z+1| < 2\}$ and $v(-1) \neq 0$.

Substituting $w(z) = (z+1)^{\rho} v(z)$ in equation (0.1) and comparing coefficients we obtain the indicial equation $\rho(\rho-1) + \rho\frac{p(z)}{1-z} + \frac{q(z)}{(1-z)^2} = 0$ at $z = -1$, hence $\rho = \pm\frac{\sqrt{k}}{2}$. Therefore, $u_1(z) = (1+z)^{\sqrt{k}/2}L(z)$ and $v_1(z) = (1+z)^{-\sqrt{k}/2}T(z) + cu_1(z)\log(1+z)$ with L, T analytic in $\{z \in \mathbf{C} : |z+1| < 2\}$ and both $L(-1)$, $T(-1) \neq 0$, are fundamental solutions of equation (0.1) in $\{z \in \mathbf{H} : |z+1| < 2\}$. Note that $c = 0$ if $k \neq m^2, m \in \mathbf{Z}_+$ and $c = 1$ if $k = 0$ (since the Wronskian of a fundamental matrix is a non-zero multiple of $\frac{1}{1-z^2}$).

In the next set of propositions we shall investigate the nature of the eigenvalues and the eigenfunctions of our Sturm–Liouville problem.

If λ is an eigenvalue of our problem and if ϕ_λ is a corresponding eigenfunction, then by the above analysis ϕ_λ is the restriction to $(-1, 1)$ of a solution of the extension of equation (0.1) to the complex plane which is analytic in H [2, p. 287].

Proposition 1.1. Let λ, ϕ_λ be as above, then:

(a) ϕ_λ is the unique such eigenfunction up to scalar multiplication.

(b) $\lim\limits_{\substack{z\to\pm1 \\ -1<z<1}} \phi_\lambda(z)(1-z^2)^{-\sqrt{k}/2}$ both exist and are non-zero.
In particular, ϕ_λ extends by continuity to $[-1, 1]$.

(c) $\lim\limits_{\substack{z\to\pm1 \\ -1<z<1}} \phi'_\lambda(\bar{z})\phi_\lambda(z) - \phi'_\lambda(z)\phi_\lambda(\bar{z}) = 0$

(d) $\lambda \in \mathbf{R}$.

Proof. Assertion (a) is clear and we have $\phi_\lambda(z) = (1-z)^{\sqrt{k}/2}X(z)$ for $z \in \{z \in \mathbf{H} : |z-1| < 2\}$ and $\phi_\lambda(z) = (1+z)^{\sqrt{k}/2}Y(z)$ for $z \in \{z \in \mathbf{H} : |z+1| < 2\}$ where X(resp. Y) is analytic in $\{z \in \mathbf{C} : |z-1| < 2\}$ (resp. in $\{z \in \mathbf{C} : |z+1| < 2\}$) and $X(1) \neq 0$ (resp. $Y(-1) \neq 0$) so that $\lim\limits_{\substack{z\to\pm 1\\ -1<z<1}} \phi_\lambda(z)(1-z)^{\sqrt{k}/2}$ and $\lim\limits_{\substack{z\to\pm 1\\ -1<z<1}} \phi_\lambda(z)(1+z)^{\sqrt{k}/2}$ exist and are non-zero, hence we have (b). Also $\phi'_\lambda(\bar{z})\phi_\lambda(z) - \phi'_\lambda(z)\phi_\lambda(\bar{z}) = (1-z)^{\sqrt{k}}(X'(\bar{z})X(z) - X'(z)X(\bar{z}))$ for $-1 < z < 1$ hence $\lim\limits_{z\to(\pm 1)^\mp} \phi'_\lambda(\bar{z})\phi_\lambda(z) - \phi'_\lambda(z)\phi_\lambda(\bar{z})$ exist.

Write equation (0.1) in the form $((1-z^2)w')' + \frac{q(z)}{(1-z^2)^2}w = 0$. Multiply this equation by $\bar{w}$ and its conjugate by w, subtract the resulting equations and integrate from r to $r, 0 < r < 1$, we get $(1-r^2)(\bar{w}'w - w'\bar{w})|_{-r}^{r} = (\lambda - \bar{\lambda})\int_{-r}^{r} w^2 \mathrm{d}z$. Take limit as $r \to 1^-$, then by the above we get (d). Now we may assume from equation (0.1) that ϕ_λ is real -valued for $-1 < z < 1$, hence (c) is clear. □

Now we investigate the solutions of equation (0.1) in the unit disc U $= \{z \in \mathbf{C} : |z| < 1\}$ for any $\lambda \in \mathbf{C}$. Let $w = w(z, \lambda)$ be such a solution and put $w = u(1-z^2)^{\sqrt{k}/2}$ in equation (0.1) to get

$$(1-z^2)u'' - 2(\sqrt{k}+1)zu' - (V(z) + k + \sqrt{k} - \lambda)u = 0. \tag{1.3}$$

Note that $u(0) = w(0)$, and u is analytic in U. Let $u(z) = \sum_{s\geqslant 0} a_s z^s$ in U and set $b_s = a_{2s}$ and $c_s = a_{2s+1}$ for $s \geqslant 0$. Substituting in equation (1.3) we get the following two linear difference equations for all $s > 0$

$$2s(2s-1)b_s - ((2s-2)^2 + M(2s-2) - \lambda')b_{s-1} - \sum_{m=1}^{s-1} \alpha_{2m} b_{s-m-1} = 0,$$

$$(2s+1)2sc_s - ((2s-1)^2 + M(2s-1) - \lambda')c_{s-1} - \sum_{m=1}^{s-1} \alpha_{2m} c_{s-m-1} = 0,$$

where $M = 2\sqrt{k}+1$ and $\lambda' = \lambda - \alpha_0 - k - \sqrt{k}$.

Solving this system of difference equations by Cramer's rule we get $b_s = \frac{b_0}{(2s)!}\det(B_s - I\lambda')$ and $c_s = \frac{c_0}{(2s+1)!}\det(C_s - I\lambda')$ where

$$B_s = \begin{bmatrix} \theta_s & \alpha_2 \cdots & \alpha_{2(s-1)} \\ a_{s-1} & \cdots\cdot & \cdot \\ & \cdots\cdot & \cdot \\ & \cdots & \cdot \\ & \cdot\ \cdot & \alpha_2 \\ & a_1 & \theta_1 \end{bmatrix}, \qquad C_s = \begin{bmatrix} \theta'_s & \alpha_2 \cdots & \alpha_{2(s-1)} \\ a'_{s-1} & \cdots\cdot & \cdot \\ & \cdots\cdot & \cdot \\ & \cdots & \cdot \\ & \cdot\ \cdot & \alpha_2 \\ & a'_1 & \theta'_1 \end{bmatrix},$$

$$a_i = -2i(2i-1), \qquad \theta_i = (2i-2)^2 + 2(i-1)M \text{ and}$$
$$a'_i = -2i(2i+1), \qquad \theta'_i = (2i-1)^2 + (2i-1)M.$$

Let $\beta_i^{(s)}$ (resp. $\gamma_i^{(s)}$), $1 \leqslant i \leqslant s$, be the eigenvalues of B_s (resp. C_s) arranged such that $Re\ \beta_1^{(s)} \leqslant Re\ \beta_2^{(s)} \leqslant \cdots \leqslant Re\ \beta_s^{(s)}$ (resp. $Re\ \gamma_1^{(s)} \leqslant Re\ \gamma_2^{(s)} \leqslant \cdots \leqslant Re\ \gamma_s^{(s)}$).

Let $\phi_0(z,\lambda) = (1-z^2)^{\sqrt{k}/2}\sum_{s\geqslant 0} b_s z^{2s}$, $\quad b_0 \neq 0$ and

$\phi_1(z,\lambda) = (1-z^2)^{\sqrt{k}/2} z \sum_{s\geqslant 0} c_s z^{2s}$, $c_0 \neq 0$, then ϕ_0, ϕ_1 is a fundamental system of solutions of equation (0.1) in U.

Proposition 1.2. λ eigenvalue for our problem iff either $\phi_0(z,\lambda)$ or $\phi_1(z,\lambda)$ is bounded on $(-1,1)$.

Proof. The if part is clear. Suppose that λ is an eigenvalue for our problem then we may assume that $\phi_\lambda(z) = \phi_1(z,\lambda) + b_0^{-1}\phi_\lambda(0)\phi_0(z,\lambda)$ in U. If $\phi_\lambda(0) = 0$ we are done, otherwise $\phi_0(z,\lambda) = \frac{b_0}{2\phi_\lambda(0)}(\phi_\lambda(z) + \phi_\lambda(-z))$ is bounded on $(-1,1)$. □

According to propositions 1.1 and 1.2 we may assume that the eigenfunction $\phi_\lambda(z)$ corresponding to the eigenvalue λ is a real-valued continuous function on $[-1,1]$ which is either even with $\phi_\lambda(0) > 0$ or odd with $\phi'_\lambda(0) > 0$ and that $\int_{-1}^{1} \phi_\lambda^2\, \mathrm{d}z = 1$. This determines $\phi_\lambda(z)$ uniquely. We shall make this assumption henceforth.

Now we turn to the characterization of the eigenvalues (theorem 1.5). We shall need the following lemmas.

Lemma 1.3. Suppose that $\sum_{s\geqslant 0} d_s z^{2s}$ has a radius of convergence $\geqslant 1$ and that this series is bounded on $(-1,1)$, then

1. $\liminf_{s\to\infty} s\ Re\ d_s \leqslant 0$, $\quad \limsup_{s\to\infty} s\ Re\ d_s \geqslant 0$ and
2. $\liminf_{s\to\infty} s\ \ \mathrm{Im}\ d_s \leqslant 0$, $\quad \limsup_{s\to\infty} s\ \mathrm{Im}\ d_s \geqslant 0$.

In particular, if $\lim_{s\to\infty} s\ d_s$ exists it must equal to zero.

Proof. Clearly we may assume d_s real for all s. We will show that $\liminf_{s\to\infty} s\ d_s \leqslant 0$ and a similar proof holds for the lim sup result.

Suppose $\liminf_{s\to\infty} s\ d_s > 0$, then there exists some $c > 0$ such that $d_s \geqslant s^{-1}c$ for $s \geqslant s_0 \geqslant 1$. Let $M_{s_0} = \sup_{-1\leqslant z\leqslant 1}\left|\sum_{s=0}^{s_0-1} d_s z^{2s}\right|$ and $\sup_{-1<z<1}\left|\sum_{s\geqslant 0} d_s z^{2s}\right| \leqslant$ L $< \infty$ so that for all $-1 < z < 1$ we have $L \geqslant \left|\sum_{s\geqslant s_0} d_s z^{2s}\right| - M_{s_0} \geqslant c\sum_{s\geqslant s_0} s^{-1}z^{2s} - M_{s_0}$. But $\sum_{s=s_0}^{s_0+m} s^{-1} > 2c^{-1}(M_{s_0} + L)$ for some $m \geqslant 1$ so that $\sum_{s=s_0}^{s_0+m} s^{-1}z^{2s} > 2c^{-1}(M_{s_0} + L)$ for $0 \leqslant 1 - z^2 \leqslant \delta < 1$ which is absurd. □

Lemma 1.4. Let $\alpha > 0$ and suppose $f(z) = \sum_{s\geqslant 0} d_s z^{2s}$ where the series has a radius of convergence $\geqslant 1$ and $\lim_{s\to\infty} s^{1-\alpha} d_s = 0$, then $\lim_{r\to(\pm 1)^\mp}(1-r^2)^\alpha f(re^{i\theta}) = 0$ uniformly for $0 \leqslant \theta \leqslant 2\pi$.

Proof. Since $\lim_{s\to\infty} \frac{s!s^{\alpha-1}}{\alpha(\alpha+1)\dots(\alpha+s-1)} = \Gamma(\alpha)$, we have $\left|\frac{s!s^{\alpha-1}}{\alpha(\alpha+1)\dots(\alpha+s-1)}\right| \leqslant L < \infty$ for $s \geqslant 1$.

Let $\varepsilon > 0$ and let $|s^{1-\alpha}d_s| < \frac{\varepsilon}{3L}$ for $s \geqslant s_0 \geqslant 1$ and $\max_{1\leqslant s\leqslant s_0}\left|s^{1-\alpha}d_s\right| = L_1$ and $\sum_{s=1}^{s_0} \frac{\alpha(\alpha+1)\dots(\alpha+s-1)}{s!} < \frac{\varepsilon}{3LL_1}(1-r^2)^{-\alpha}$ and $\left|(1-r^2)^\alpha d_0\right| \frac{\varepsilon}{3}$ for $0 < 1-\delta < |r| < 1$ so that $|f(re^{i\theta})| = \left|d_0 + \sum_{s\geqslant 1} \frac{s!s^{\alpha-1}}{\alpha(\alpha+1)\dots(\alpha+s-1)} \cdot (\frac{\alpha(\alpha+1)\dots(\alpha+s-1)}{s!} r^{2s}) e^{2i\theta s} s^{1-\alpha} d_s\right| \leqslant \frac{\varepsilon}{3}(1-r^2)^{-\alpha} + L\left(L_1 \sum_{s=1}^{s_0} \frac{\alpha(\alpha+1)\dots(\alpha+s-1)}{s!} + \frac{\varepsilon}{3L}(1-r^2)^{-\alpha}\right) < \varepsilon(1-r^2)^{-\alpha}$ for $0 < 1-\delta < |r| < 1$ and all $0 \leqslant \theta \leqslant 2\pi$. □

Theorem 1.5. $\phi_0(z,\lambda')$ is bounded on (-1,1) iff $\lim_{s\to\infty} s^{1-\sqrt{k}} b_s(\lambda') = 0$ and $\phi_1(z,\lambda')$ is bounded on $(-1,1)$ iff $\lim_{s\to\infty} s^{1-\sqrt{k}} c_s(\lambda') = 0$.

Proof. We shall prove the first statement and the proof of the second one is similar.

$\Rightarrow$ Suppose that $\phi_0(z,\lambda')$ is bounded on $(-1,1)$ then λ' is real and $\sum_{s\geqslant 0} b_s(\lambda') z^{2s}$ is bounded on $(-1,1)$ by proposition 1.1. Now lemma 1.3 gives $\liminf_{s\to\infty} s b_s(\lambda') \leqslant 0$ and $\limsup_{s\to\infty} s b_s(\lambda') \geqslant 0$. By theorem A.8 $\lim_{s\to\infty} s^{1-\sqrt{k}} b_s(\lambda')$ exists so that if $\lim_{s\to\infty} s^{1-\sqrt{k}} b_s(\lambda') > 0$ then $b_s(\lambda') > 0$ for $s \geqslant s_0$ and $0 < \lim_{s\to\infty} s^{1-\sqrt{k}} b_s(\lambda') \leqslant \liminf_{s\to\infty} s b_s(\lambda') \leqslant 0$ which is absurd, and if $\lim_{s\to\infty} s^{1-\sqrt{k}} b_s(\lambda') < 0$ then there exists some $c > 0$ such that $s^{1-\sqrt{k}} b_s(\lambda') < -c$ for $s \geqslant s_1$ and $s b_s(\lambda') < -cs^{\sqrt{k}} \leqslant -c$ for $s \geqslant s_1$ and $\limsup_{s\to\infty} s b_s(\lambda') \leqslant -c < 0$ which is absurd also.

$\Leftarrow$ Suppose $\lim_{s\to\infty} s^{1-\sqrt{k}} b_s(\lambda') = 0$ then for $k > 0$ lemma 1.4 gives $\lim_{r\to(\pm 1)^\mp}(1-r^2)^{\sqrt{k}} \sum_{s\geqslant 0} b_s(\lambda') r^{2s} = 0$. Therefore, $(1-z^2)^{\sqrt{k}} \sum_{s\geqslant 0} b_s(\lambda') z^{2s} = (1-z^2)^{\sqrt{k}/2} \phi_0(z,\lambda')$ and $\phi_0(z,\lambda')$ has no unbounded terms near $z = \pm 1$. If $k = 0$, let $\sum_{s\geqslant 0} b_s(\lambda') z^{2s} = \alpha_1 L(z) + \alpha_2(M(z) + L(z)\log(1+z))$ in U. Differentiating we get

$$\frac{2}{z}\sum_{s\geqslant 0} s b_s(\lambda') z^{2s} = \alpha_1 L'(z) + \alpha_2\left(M'(z) + \frac{L(z)}{1+z} + L'(z)\log(1+z)\right)$$

and since $\lim_{s\to\infty} s b_s(\lambda') = 0$, lemma 1.4 gives $\lim_{r\to(-1)^+}(1-z^2)\sum_{s\geqslant 0} s b_s(\lambda') z^{2s} = 0$ hence $\alpha_2 = 0$ and $\phi_0(z,\lambda')$ is bounded near $z = -1$ and, by a similar argument, near $z = +1$ as well. □

2. Computation of the eigenvalues and eigenfunctions

In this section, we follow [3, p. 1473] and show that the eigenvalues form a strictly increasing sequence of real numbers $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$ such that $\lim_{n\to\infty} \lambda_n = \infty$ and that ϕ_{λ_n} has precisely n zeros in $(-1, 1)$, theorem 2.4. This will enable us to formulate our final solution of the problem in theorem 2.7.

Let $w_{-1}(z, \lambda)$ be a bounded solution of equation (0.1) near-1. Note that $w_{-1}(z, \lambda)$ has only finitely many zeros in $(-1, 1)$, all are simple, since otherwise these zeros would accumulate at +1 and $w_{-1}(z, \lambda)$ cannot coincide near $+1$ with any bounded or unbounded solution of equation (0.1). w we may also assume that $w_{-1}(z, \lambda)$ is real-valued on (−1,1) for λ real.

Let $J_n = \{\lambda \in \mathbf{R} : w_{-1}(z, \lambda)$ has exactly n zeros in $(-1, 1)\}$ and write equation (0.1) } in the form

$$((1 - z^2)w')' + q_1(z)w = 0, \tag{2.1}$$

where q_1(z)=$\frac{q(z)}{(1-z^2)^2}$.

Proposition 2.1. Let $\lambda_1 < \lambda_2; \lambda_1, \lambda_2 \in \mathbf{R}$ and let x_1 be the smallest zero of $w_{-1}(z, \lambda_1)$ in (−1,1),then $w_{-1}(z, \lambda_2)$ has a zero in $(-1, x_1)$ and the nth zero of $w_{-1}(z, \lambda_2)$ is less than the nth zero of $w_{-1}(z, \lambda_1)$ in (−1,1).

Proof. We shall prove the first statement. The second one follows from it and the comparison theorem [1, p. 434].

Let $w_i = w_{-1}(z, \lambda_i)$, $i = 1, 2$ and suppose w_1 and $w_2 > 0$ on $(-1, x_1)$. Multiply equation (2.1) for w_1 by w_2 and the same equation for w_2 by w_1, subtract and integrate from r to x_1, $-1 < r < x_1$, we get $(1 - z^2)(w_1' w_2 - w_2' w_1)|_r^{x_1} = (\lambda_2 - \lambda_1)\int_r^{x_1} w_1 w_2 \, dz$. Note that $\lim_{z\to(-1)^+} (1 - z^2)w_1' w_2 = \lim_{z\to(-1)^+} (1 - z^2)w_2' w_1$ so that $(1 - x_1^2)w_1'(x_1)w_2(x_1) = (\lambda_2 - \lambda_1)\int_{-1}^{x_1} w_1 w_2 \, dz > 0$ while the L.H.S. is $\leqslant 0$ which is absurd. □

Corollary 2.2.

1. $J_0 \neq \varnothing$.
2. $\{n \in \mathbf{Z}_+ : J_n \neq \varnothing\}$ is unbounded above.
3. $\lambda_1 \in J_n$, $\lambda_2 \in J_m$, $n < m \Rightarrow \lambda_1 < \lambda_2$ so that J_n is an interval in $\mathbf{R}$.
4. J_n is a left open and right closed interval in $\mathbf{R}$ for all $n \geqslant 0$.
5. J_n contains at most one eigenvalue for all $n \geqslant 0$.

Proof. 1. Suppose that $J_0 = \emptyset$ and let x_1 be the smallest zero of $w = w_{-1}(z,\lambda)$ for $\lambda < \min_{-1\leqslant x\leqslant 1} V(x)$ in (−1,1). Multiply equation (2.1) by w and integrate from r to x_1, $-1 < r < x_1$, we get $(1-z^2)w'w|_r^{x_1} \geqslant (\min_{-1\leqslant x\leqslant 1} V(x) - \lambda)\int_r^{x_1} w^2\mathrm{d}z$. As $r \to (-1)^+$we get $\int_r^{x_1} w^2\,\mathrm{d}z \leqslant 0$ which is absurd.

2. Let $k < L \in \mathbf{Z}_+$, then for all $n \in \mathbf{Z}_+$, $q_1(z,\lambda) > -\frac{L^2}{L-z^2} + (n+L)(n+L+1)$ for $\lambda \geqslant \lambda_* = \lambda_*(n)$. Note that $w = (1-z^2)^{\frac{L}{2}} \frac{\mathrm{d}^L}{\mathrm{d}z^L}(P_{n+L+1}(z))$ is a solution for $((1-z^2)w')' + \left(-\frac{L^2}{1-z^2} + (n+L+1)(n+L+2)\right) w = 0$, where $P_{n+L+1}(z)$ is the Legendre polynomial of degree $n+L+1$, so that w has $n+1$ zeros in (−1, 1) and $w_{-1}(z,\lambda)$ has at least n zeros in (−1, 1) for $\lambda \geqslant \lambda_*$ by the comparison theorem [1, p. 434].

3. Suppose $\lambda_1 \in J_n$, $\lambda_2 \in J_m$, $n < m$ and $\lambda_2 < \lambda_1$ then by proposition 2.1 $w_{-1}(z,\lambda_1)$ has at least m zeros in (−1, 1) which is absurd.

4. It suffices to prove that J_n is right closed for all $n \geqslant 0$. Let $\lambda = \sup J_n < \infty$ by 2 + 3. Suppose $\lambda \notin J_n$ so that $w_{-1}(z,\lambda_1)$ has at least $n+1$ zeros in (−1, 1). Since $w_{-1}(z,\lambda)$ is analytic in U×**C** [2, p. 299] and $(w_{-1}(z,\lambda))^{-} = w_{-1}(\overline{z},\lambda)$ for $z \in$ U and $\lambda \in \mathbf{R}$ then for $\lambda' \in \mathbf{R}$, $|\lambda-\lambda'| < r$ for some $r > 0$, $w_{-1}(z,\lambda')$ would have at least $n+1$ zeros in(−1,1) [2, p. 248] which is absurd.

5. Suppose λ_1, λ_2 eigenvalues in some J_n, $n \geqslant 0$, $\lambda_1 < \lambda_2$ and let w_i, $i = 1, 2$, be the corresponding eigenfunctions. If $n = 0$, w_i are even solutions by proposition 1.2 for $i = 1, 2$, hence $(1-z^2)(w_1'w_2 - w_2'w_1)|_{-1}^0 = (\lambda_2-\lambda_1)\int_{-1}^0 w_1w_2\,\mathrm{d}z$ gives $(\lambda_2-\lambda_1)\int_{-1}^0 w_1w_2\,\mathrm{d}z = 0$ which is absurd. If $n > 0$, let $-1 < x_1 \leqslant 0$ be the least zero of w_1 in(−1, 1) then, whether we have even or odd eigenfunction according to the parity of n, by proposition 1.2, w_2 would have at least $n+1$ zeros in (−1, 1) by proposition 2.1 + the comparison theorem [1, p. 434], which is absurd. □

Now for all $s \geqslant 1$, define $\nu_s : \bigcup_{n\geqslant s} J_n \to (-1,1)$ by $\nu_s(\lambda)$ = the sth zero of $w_{-1}(z,\lambda)$ in (−1, 1).

Lemma 2.3. For all $\geqslant 1$

1. ν_s is a strictly decreasing continuous function.
2. Let $\lambda_{s-1} = \inf \bigcup_{n\geqslant s} J_n$, then $\lim_{\lambda\to(\lambda_{s-1})^+} \nu_s(\lambda) = +1$ and $J_s \neq \emptyset$.
3. $\lim_{\lambda\to\infty} \nu_s(\lambda) < 0$.

Proof. 1. Clearly ν_s is strictly decreasing by proposition 2.1 and the analyticity of $w_{-1}(z,\lambda)$ on U×**C** [2, p. 299] shows that ν_s is upper semicontinuous by [2, p. 248]. Let $\lambda_0 \in \bigcup_{n\geqslant s} J_n$ and $\lambda_0 < \lambda$, then for all $\lambda' < \lambda$ we have $\nu_1(\lambda') > \nu_1(\lambda)$ by proposition 2.1, hence [2, p. 248] shows further that ν_s is lower semicontinuous.

2. If $s = 1$ suppose that $\lim_{\lambda\to(\lambda_0)^+} \nu_1(\lambda) = a < 1$ hence $w_{-1}(a,\lambda_0) = 0$ and $\lambda_0 = \sup J_0$ which is absurd.

   Let $\lambda_0 < \lambda$ and let x be the largest zero of $w_{-1}(z,\lambda)$ in $(-1,1)$, then there exists $\lambda_0 < \lambda_1 < \lambda$ such that $x < \nu_1(\lambda_1) < 1$ and $\lambda_1 \in J_1$ by the comparison theorem [1, p. 434]. Assume by induction that $\lim_{\lambda\to(\lambda_{s-1})^+} \nu_s(\lambda) = +1$ and $J_s \neq \emptyset$ for $s \leqslant s_0$, $s_0 \geqslant 1$. If $\lim_{\lambda\to(\lambda_{s_0})^+} \nu_{s_0+1}(\lambda) = a < 1$, then $\lambda_{s_0} = \sup J_{s_0}$ and $w_{-1}(a,\lambda_{s_0}) = 0$ and $\nu_{s_0+1}(\lambda) < \nu_{s_0}(\lambda_{s_0})$ for all $\lambda > \lambda_{s_0}$ since otherwise $\nu_{s_0}(\lambda_{s_0}) \leqslant \nu_{s_0+1}(\lambda)$ for some $\lambda > \lambda_{s_0}$ gives $\nu_{s_0}(\lambda_{s_0}) < a$ and $\lambda_{s_0} \in J_n$ for some $n \geqslant s_0 + 1$ which is absurd.

   Let $m < \min(\nu_1(\lambda_{s_0}) + 1, 1 - \nu_{s_0}(\lambda_{s_0}))$, then by 1+[2, p. 248] there exists some $r > 0$ such that for all $\lambda_{s_0} \leqslant \lambda' < \lambda_{s_0} + r$ all the zeros of $w_{-1}(z,\lambda')$ in $(-1, \nu_{s_0}(\lambda_{s_0}) + m)$ belong to $(\nu_1(\lambda_{s_0}) - m, \nu_{s_0}(\lambda_{s_0}) + m)$ and they are exactly s_0 in number which is absurd for $\lambda_{s_0} < \lambda$ by the above.

   Now the same argument as in the $s = 1$ case shows that $J_{s_0+1} \neq \emptyset$.

3. Let $k < L \in \mathbf{Z}_+$ and note that $\frac{d^{2L}}{dz^{2L}} P_{2n+2L+2}(z)$ has $n + 1$ zeros in $(-1, 0)$, hence, by the comparison theorem [1, p. 434], $w_{-1}(z,\lambda)$ for $\lambda \geqslant \lambda_*$ has at least n zeros in $(-1,0)$ as in corollary 2.2 part 3 and $\nu_n(\lambda) > 0$ for $\lambda \geqslant \lambda_*$.

□

Theorem 2.4. The eigenvalues form a strictly increasing sequence of real numbers $\lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots$ such that $\lim_{n\to\infty} \lambda_n = \infty$ and the corresponding eigenfunction ϕ_{λ_n} has exactly n zeros in $(-1, 1)$ for all $n \geqslant 0$.

Proof. It suffices by corollary 2.2 and Lemma 2.3 to show that each J_n contains an eigenvalue.

By lemma 2.3 for all $s \geqslant 1$ there exists some λ_s^* such that $\nu_s(\lambda_s^*) = 0$ and $w_{-1}(0,\lambda_s^*) = 0$ so that $w_{-1}(z,\lambda_s^*) = \phi_1(z,\lambda_s^*)$ and λ_s^* is an eigenvalue. Note that $w_{-1}(z,\lambda_s^*)$ has $2s-1$ zeros in $(-1,1)$ and $\lambda_s^* \in J_{2s-1}$. Now for $s \geqslant 1$ let λ_s^* and λ_{s+1}^*, be the eigenvalues in J_{2s-1} and J_{2s+1}, respectively, then $w_{-1}(z,\lambda_s^*) = w_{-1}(z,\lambda_{s+1}^*) = 0$ hence there exists some $\lambda_s^* < \lambda < \lambda_{s+1}^*$ such that $w'_{-1}(0,\lambda) = 0$ and $w_{-1}(z,\lambda) = \phi_0(z,\lambda)$ and λ eigenvalue $\in J_{2s}$.

Suppose that $w'_{-1}(0,\lambda)$ has no zeros for $\lambda \leqslant \lambda_1^*$ and let $w'_{-1}(0,\lambda) > 0$ for $\lambda \leqslant \lambda_1^*$, then since $w_{-1}(0,\lambda_1^*) = 0$ we have $w'_{-1}(0,\lambda)w_{-1}(0,\lambda) < 0$ for $\lambda \leqslant \lambda_1^*$. Let $w = w_{-1}(z,\lambda)$ and multiply equation (2.1) by w and integrate from r to 0, $-1 < r < 0$, we get $(1-z^2)w'w|_r^0 \geqslant \left(\min_{-1\leqslant x\leqslant 1} V(x) - \lambda\right)\int_r^0 w^2\,\mathrm{d}z$ and as $r \to (-1)^+$ we get $w'(0)w(0) \geqslant \left(\min_{-1\leqslant x\leqslant 1} V(x) - \lambda\right)\int_{-1}^0 w^2\,\mathrm{d}z > 0$ for $\lambda < \min_{-1\leqslant x\leqslant 1} V(x)$ which is absurd if $\lambda \leqslant \lambda_1^*$. □

Remark 2.5. The eigenfunctions (ϕ_λ) actually form a Hilbert basis of $L^2([-1,1])$. This can be seen as follows.

Let $D = \{w \in C^2((-1,1)) : w, (1-x^2)w', (1-x^2)^2w''$ all extend by continuity to $[-1,1]\}$ and define the linear operator $L : D \to C([-1,1])$ by

$$Lw = [((1-x^2)w')' - (k(1-x^2)^{-1} + V(x) - \lambda_0 + 1)w](1-x^2).$$

Note that $\ker L = 0$ and that D is dense in $L^2([-1,1])$.

Let $w_{-1}(x,\lambda)$ (respectively, $w_1(z,\lambda)$) denote a bounded solution of equation (0.1) near -1 (respectively, $+1$) and construct the Green function

$$G(x,y) = \left\{\begin{array}{ll} cw_{-1}(x,\lambda_0-1)w_1(y,\lambda_0-1), & x \leqslant y \leqslant 1 \\ cw_{-1}(y,\lambda_0-1)w_1(x,\lambda_0-1) & -1 \leqslant y \leqslant x \end{array}\right\},$$

where $(1-x^2)(w_1'(x,\lambda_0-1)w_{-1}(x,\lambda_0-1) - w_1(x,\lambda_0-1)w'_{-1}(x,\lambda_0-1))$ =non-zero constant= c^{-1}.

Observe that $k_{\pm 1}(\lambda) = \displaystyle\int_{-1}^1 \left|\frac{(1\pm y)^{\sqrt{k}/2}w_{\pm 1}(y,\lambda)}{(1\mp y)^{\sqrt{k}/2}}\right|^2 \mathrm{d}y < \infty$ and let

$$\begin{aligned} k(\lambda) = \max(k_{-1}(\lambda), k_1(\lambda)) \text{ so that } & \int_{-1}^1 \mathrm{d}x \int_{-1}^1 |G(x,y)|^2 \mathrm{d}y \\ = c^2 \int_{-1}^1 \mathrm{d}x\Big[&\int_{-1}^x |w_{-1}(y,\lambda_0-1)w_1(x,\lambda_0-1)|^2\mathrm{d}y \\ &+ \int_x^1 |w_{-1}(x,\lambda_0-1)w_1(y,\lambda_0-1)|^2\mathrm{d}y\Big] \\ &\leqslant c^2k(\lambda_0-1) \\ &\times \int_{-1}^1 \left[\left|\frac{(1+x)^{\sqrt{k}/2}w_1(x,\lambda_0-1)}{(1-x)^{\sqrt{k}/2}}\right|^2 + \left|\frac{(1-x)^{\sqrt{k}/2}w_{-1}(x,\lambda_0-1)}{(1+x)^{\sqrt{k}/2}}\right|^2\right]\mathrm{d}x \\ &\leqslant 2c^2(k(\lambda_0-1))^2 < \infty \end{aligned}$$

and G defines a compact operator, which we also denote by G, on $L^2([-1,1])$ via $f \to \int_{-1}^1 G(x,y)f(y)\,\mathrm{d}y$. Note that if f is a continuous function with compact support in $(-1,1)$, then $G(f) \in D$ hence $G(L^2([-1,1]) \subseteq D$ [2, p. 318] and for all $f \in L^2([-1,1])$ we have $LG(f) = (1-x^2)f$ and $\ker G = 0$.

Now λ eigenvalue with eigenfunction ϕ_λ iff $L\phi_\lambda = (\lambda_0 - 1 - \lambda)(1 - x^2)\phi_\lambda$ iff $(\lambda_0 - 1 - \lambda)^{-1}$ eigenvalue of G with eigenfunction ϕ_λ hence (ϕ_λ) form a Hilbert basis of $L^2([-1, 1])$ [2, p. 331].

Proposition 2.6. For all $i \geqslant 1$, $\lim_{i \leqslant s \to \infty} \beta_i^{(s)}$ (respectively, $\lim_{i \leqslant s \to \infty} \gamma_i^{(s)}$) exists and $\{\lambda'_{2n} : n \geqslant 0\} = \left\{ \lim_{i \leqslant s \to \infty} \beta_i^{(s)} : i \geqslant 1 \right\} \left(\text{respectively, } \{\lambda'_{2n+1} : n \geqslant 0\} = \left\{ \lim_{i \leqslant s \to \infty} \gamma_i^{(s)} : \mathrm{i} \geqslant 1 \right\} \right)$.

Proof. We will prove the statement about the β's and the statement about the γ's is proved similarly.

Note that if $\lim_{i \leqslant s \to \infty} \beta_i^{(s)}$ exists for all these must be even eigenvalues by theorem's 1.5 and A.8 and we have $\lim_{i \leqslant s \to \infty} \beta_i^{(s)} \leqslant \lim_{i \leqslant s \to \infty} \beta_j^{(s)}$ for $1 \leqslant i \leqslant j$ so that an even eigenvalue is really $\lim_{i \leqslant s \to \infty} \beta_i^{(s)}$ for some $i \geqslant 1$ by theorems A.8 and 1.5 again, hence it suffices to prove that $\lim_{i \leqslant s \to \infty} \beta_i^{(s)}$ exists for all $i \geqslant 1$.

For $i = 1$ let λ'_0 be the minimum eigenvalue by theorem 2.4 so that by Theorems 1.5 and A.8 for all $r > 0$, $|\lambda'_0 - \beta_j^{(s)}| < r$ for some $1 \leqslant j \leqslant s$ and for all $s \geqslant s_0 \geqslant 1$, hence $|\lambda'_0 - \beta_j^{(s)}| < r$ for all $s \geqslant s_1 \geqslant s_0$ since otherwise we have $(2(i-1))^2 + 2(i-1)M - 2(i-1)(2i-3) - \sum_{j \geqslant 1} |\alpha_{2j}| > \lambda'_0$ for $i \geqslant i_0$ and Gershgorin theorem shows that for some subsequence s_m, $\beta_1^{(s_m)}$ converge to x where $|x - \lambda'_0| \geqslant r_0 > 0$ and $Rex \leqslant \lambda'_0$ so that by theorem's 1.5 and A.8 and Proposition 1.1 we get $x \leqslant \lambda'_0 - r_0 \leqslant \lambda'_0$ which is absurd by choice of λ'_0.

Assume by induction that $\lim_{i \leqslant s \to \infty} \beta_i^{(s)}$ exists for $i \leqslant i_0, i_0 \geqslant 1$ and let $m = \left| \left\{ \lim_{i \leqslant s \to \infty} \beta_i^{(s)} : i \leqslant i_0 \right\} \right|$ so that $\{\lambda'_{2n} : 0 \leqslant n \leqslant m-1\} = \left\{ \lim_{i \leqslant s \to \infty} \beta_i^{(s)} : i \leqslant i_0 \right\}$. theorems 1.5 and A.8 and Induction hypothesis show that for all $0 \leqslant r \leqslant r_1$, $\left| \lambda'_{2m} - \beta_j^{(s)} \right| < r$ for some $i_0 + i \leqslant j \leqslant s$ for all $s \geqslant s_0 \geqslant i_0 + 1$. As above, Gershgorin theorem shows that $\left\{ \beta_{i_0+1}^{(s)} : i_0 + 1 \leqslant s \right\}$ has two possible accumulation points $\lambda'_{2(m-1)}$ and λ'_{2m}. Suppose that $\liminf_{i_0+1 \leqslant s \to \infty} \beta_{i_0+1}^{(s)} = \lambda'_{2(m-1)} < \lambda'_{2m} = \limsup_{i_0+1 \leqslant s \to \infty} \beta_{i_0+1}^{(s)}$ and let $\lambda'_{2(m-1)} = a(a+1) - k - \sqrt{k}$ so that a is a zero of the entire function $\lim_{s \to \infty} s^{1-\sqrt{k}} b_s(a)$ of multiplicity $n(a) \geqslant 1$ by theorem A.8. Now the proof of proposition A.7 shows that there exists some $r_2 > 0$ such that for all $0 < r \leqslant r_2$, $|\{1 \leqslant j \leqslant s : |\lambda'_{2(m-1)} - \beta_j^{(s)}| < r\}| = n(a)$ for all $s \geqslant s_1 \geqslant s_0$ and $\lim_{i_0+1 \leqslant s \to \infty} \beta_{i_0+1}^{(s)} = \lambda'_{2(m-1)}$ which is absurd. □

Now theorem 2.4 and remark 2.5 and proposition 2.6 give the solution of our Sturm–Liouville problem.

Theorem 2.7. The eigenvalues of the Sturm–Liouville problem are:

1. Eigenvalues for even solutions

$$\lambda = \alpha_0 + k + \sqrt{k} + \lim_{i \leqslant s \to \infty} \beta_i^{(s)} \quad \text{, for } i \geqslant 1.$$

2. Eigenvalues for odd solutions

$$\lambda = \alpha_0 + k + \sqrt{k} + \lim_{i \leqslant s \to \infty} \gamma_i^{(s)} \quad \text{, for } i \geqslant 1.$$

The eigenfunctions (ϕ_λ) form a Hilbert basis of $L^2([-1, 1])$. □

Appendix A

In this appendix, we put $\lambda - \alpha_0 = a(a+1)$ in equation (0.1) so that the b's and the c's are polynomials in a and our purpose is to show that the two sequence of polynomials $\{s^{1-\sqrt{k}} b_s(a) : s \geqslant 1\}$ and $\{s^{1-\sqrt{k}} c_s(a) : s \geqslant 1\}$ converge uniformly on compact sets in the a-plane to non-trivial entire transcendental functions. We also determine the zero sets of these functions. This will enable us to formulate theorem A.8. Let

$$P_s(a) = \frac{1}{(2s)!} \prod_{t=0}^{s-1} ((2t)^2 + 2tM - \lambda') \text{and}$$
$$Q_s(a) = \frac{1}{(2s+1)!} \prod_{t=0}^{s-1} ((2t+1)^2 + (2t+1)M - \lambda') \text{ so that we can write}$$
$$s^{1-\sqrt{k}} b_s(a) = s^{1-\sqrt{k}} P_s(a) R_s(a) \quad \text{and} \quad s^{1-\sqrt{k}} c_s(a) = s^{1-\sqrt{k}} Q_s(a) S_s(a).$$

Proposition A.1.

$$\lim_{s\to\infty} s^{1-\sqrt{k}} P_s(a) = \frac{\sqrt{\pi}}{2\Gamma(1/2(\sqrt{k}-a))\Gamma(1/2(\sqrt{k}+a+1))}$$

and

$$\lim_{s\to\infty} s^{1-\sqrt{k}} Q_s(a) = \frac{\sqrt{\pi}}{2\Gamma(1/2(\sqrt{k}-a+1))\Gamma(1/2(\sqrt{k}+a+2))}.$$

Proof. We shall prove the first formula and the second one is proved similarly.

$$s^{1-\sqrt{k}}P_s(a) = \frac{-\lambda' s^{1-\sqrt{k}}}{(2s)!}\prod_{t=1}^{s-1} 4t^2\left(1+\frac{\sqrt{k}-a}{2t}\right)\left(1+\frac{\sqrt{k}+a+1}{2t}\right)$$
$$= \frac{2^s(s-1)!\sqrt{s-1}}{1.3.5\ldots(2s-1)}.\frac{-\lambda'}{4}\prod_{t=1}^{s-1}\left(1+\frac{\sqrt{k}-a}{2t}\right)\mathrm{e}^{-(\sqrt{k}-a)/2t}$$

$$.\prod_{t=1}^{s-1}\left(1+\frac{\sqrt{k}+a+1}{2t}\right)\mathrm{e}^{-(\sqrt{k}+a+1)/2t}.\mathrm{e}^{\frac{1}{2}M\left(1+\frac{1}{2}+\cdots+\frac{1}{s-1}-\log(s-1)\right)}.\left(\frac{s-1}{s}\right)^{\sqrt{k}}$$
$$\to \Gamma\left(\frac{1}{2}\right).\frac{-\lambda'}{4}.\frac{\mathrm{e}^{-1/2\gamma M}}{\Gamma(1/2(\sqrt{k}-a))\Gamma(1/2(\sqrt{k}+a+1))}.\frac{4}{(\sqrt{k}-a)(\sqrt{k}+a+1)}.\mathrm{e}^{1/2\gamma M}$$

as $s\to\infty$ where γ is the Euler constant, hence our formula. □

Proposition A.2. If $a \in L$ compact $\subseteq \mathbf{C}-\{\sqrt{k}+2t, -\sqrt{k}-2t-1 : t\in\mathbf{Z}_+\}$ then $\sum_{j=2}^{\infty}\sum_{i=1}^{j-1}\left|\frac{\alpha_{2i}}{\theta_j-\lambda'}.\prod_{t=j-i}^{j-1}\frac{2t(2t-1)}{\theta_t-\lambda'}\right| \leqslant m_L$ for all $a\in L$ and If $a\in L'$ compact $\subseteq \mathbf{C}-\left\{\sqrt{k}+2t+1, -\sqrt{k}-2t-2 : t\in\mathbf{Z}_+\right\}$ then $\sum_{j=2}^{\infty}\sum_{i=1}^{j-1}\left|\frac{\alpha_{2i}}{\theta'_j-\lambda'}.\prod_{t=j-i}^{j-1}\frac{2t(2t+1)}{\theta'_t-\lambda'}\right| \leqslant m'_{L'}$ for all $a\in L'$.

Proof. We shall prove the first statement and the second one is proved similarly. Let $r=\sup\{|a| : a\in L\}<\infty$ and note that $|2(t-1)+\sqrt{k}-a|$ and $|2(t-1)+\sqrt{k}+a+1|\geqslant \alpha_L>0$ for all $a\in L$ and $t\geqslant 1$.

Now $\frac{1}{|\theta_j-\lambda'|} \leqslant \frac{1}{(2(j-1))^2}.\frac{1}{|1+\frac{M}{2(j-1)}-\frac{|\lambda'|}{(2(j-1))^2}|} \leqslant \frac{1}{2(j-1)^2}$ for $j\geqslant j_L > 1+\sqrt{1/2(r(r+1)+\sqrt{k}+k)}$.

Also $\dfrac{2t(2t-1)}{|\theta_t-\lambda'|} \leqslant \dfrac{2t(2t-1)}{|(2(t-1)+\sqrt{k}-a)(2(t-1)+\sqrt{k}+a+1)|}$

$\leqslant \dfrac{1}{|1-|\frac{\sqrt{k}-a-2}{2t}||.|1-|\frac{\sqrt{k}+a}{2t-1}||} < 4$ for $t>t_L>\sqrt{k}+r+2$ so that

$\dfrac{2t(2t-1)}{|\theta_t-\lambda'|} \leqslant \max\left\{4, \frac{2t_L(2t_L-1)}{\alpha_L^2}\right\} = K_L$ and

$$\sum_{j=2}^{\infty}\sum_{i=1}^{j-1}\left|\frac{\alpha_{2i}}{\theta_j-\lambda'}.\prod_{t=j-i}^{j-1}\frac{2t(2t-1)}{\theta_t-\lambda'}\right| \leqslant \sum_{j=2}^{\infty}\frac{1}{|\theta_j-\lambda'|}\sum_{i=1}^{j-1}|\alpha_{2i}|\mathrm{K}_{\mathrm{L}}^{\mathrm{i}}$$

$$\leqslant K'_L \sum_{j=2}^{\infty} \frac{1}{|\theta_j - \lambda'|} \text{ since } V \text{ is entire}$$

$$\leqslant K'_L \left(\frac{j_L}{\alpha_L^2} + \sum_{j \geqslant j_L} \frac{1}{2(j-1)^2} \right) \leqslant K'_L \left(\frac{j_L}{\alpha_L^2} + \frac{\pi^2}{12} \right) \leqslant m_L \text{ for all } a \in L.$$

We shall need the following lemma. □

Lemma A.3. Let $\{a_s : s \geqslant 1\}$ and $\{{}_ic_j : i, j \geqslant 1\} \subseteq \mathbf{C}$ and

$$A_s = \begin{bmatrix} 1 & {}_1c_{s-1} & {}_2c_{s-2} & \cdot & \cdot & {}_{s-1}c_1 \\ a_{s-1} & 1 & \cdot & \cdot & \cdot & \cdot \\ & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & \cdot & \cdot & \cdot & {}_2c_1 \\ & & & \cdot & 1 & {}_1c_1 \\ & & & & a_1 & 1 \end{bmatrix}.$$

1. Let $D_s(x)$ be the characteristic polynomial of $A_s, s \geqslant 2$, then $D_s(x)$ satisfies the following recurrence formula for $s \geqslant 2$

$$D_s(x) = (1-x)D_{s-1}(x) + \sum_{i=1}^{s-1} {}_ic_{s-i}(-a_{s-i})(-a_{s-i+1})\dots(-a_{s-1})$$
$$D_{s-i-1}(x) \text{ and}$$
$$D_0(x) = 1, D_1(x) = 1 - x.$$

2. Let $D_s = \det A_s$, then
   (a) $|D_s| \leqslant \prod_{j=2}^{s} \left(1 + \sum_{i=1}^{j-1} |{}_ic_{j-i}a_{j-i}a_{j-i+1}\dots a_{j-1}|\right)$.
   (b) If $\sum_{j=2}^{\infty} \sum_{i=1}^{j-1} |{}_ic_{j-i}a_{j-i}a_{j-i+1}\dots a_{j-1}| < \infty$ then $\lim_{s\to\infty} D_s$ exists.

Proof. 1. Laplace expansion via the first row of $D_s(x) = \det(A_s - Ix)$ gives the result.

2. Observe that every term in the expansion of D_s occurs with the correct sign in the expansion of $\prod_{j=2}^{s} \left(1 + \sum_{i=1}^{j-1} {}_ic_{j-i}(-a_{j-i})(-a_{j-i+1})\dots(-a_{j-1})\right)$ hence (a) follows.

Now suppose that $\sum_{j=2}^{\infty} \sum_{i=1}^{j-1} |{}_ic_{j-i}a_{j-i}a_{j-i+1}\dots a_{j-1}| < \infty$, then

$\lim_{s\to\infty}\prod_{j=2}^{s}\left(1+\sum_{i=1}^{j-1}|_i c_{j-i}a_{j-i}a_{j-i+1}\dots a_{j-1}|\right)$ exists. Therefore $|D_{n+m}-D_n|\leqslant\prod_{j=2}^{n+m}\left(1+\sum_{i=1}^{j-1}|_i c_{j-i}a_{j-i}a_{j-i+1}\dots a_{j-1}|\right)$ $-\prod_{j=2}^{n}\left(1+\sum_{i=1}^{j-1}|_i c_{j-i}a_{j-i}a_{j-i+1}\dots a_{j-1}|\right)<\varepsilon$ for $n\geqslant n_0, m\geqslant 1$, hence $\lim_{s\to\infty}D_s$ exists. □

Proposition A.4. $R(a)=\lim\limits_{s\to\infty}R_s(a)$ (respectively, $S(a)=\lim\limits_{s\to\infty}S_s(a)$) is a meromorphic function with at most simple poles at $E=\{\sqrt{k}+2t,-\sqrt{k}-2t-1 : t\in\mathbf{Z}_+\}$ (respectively, at $F=\{\sqrt{k}+2t+1,-\sqrt{k}-2t-2 : t\in\mathbf{Z}_+\}$).

Proof. We shall prove the first statement and the second one is proved similarly.

Proposition A.2, Lemma A.3 part 2b and the definition of $R_s(a)$ show that $\lim_{s\to\infty}R_s(a)$ exists for all a$\notin E$. Also for all L compact $\subseteq \mathbf{C}-E$, proposition A.2 and lemma A.3 part 2a give $|R_s(a)|\leqslant \exp m_L$ for $a\in L$, hence the R_s's converge uniformly on compact sets in $\mathbf{C}-$E to an analytic function there and it suffices to show that $\frac{1}{2\pi i}\int_{\sigma_x}(a-x)^r R(a)\,\mathrm{d}a=0$ for all $r\in\mathbf{Z}_{>1}$ and $x\in E$, where σ_x is the loop $\theta\to x+\rho e^{i\theta}$, $0<\rho<1/2$. Observe that for all $r\geqslant 1$, $(a-x)^r R_s(a)$ is analytic in $\{a\in\mathbf{C} : |a-x|<1\}$ for $x\in E$, hence $\frac{1}{2\pi i}\int_{\sigma_x}(a-x)^r R(a)\,\mathrm{d}a=\lim\limits_{s\to\infty}\frac{1}{2\pi i}\int_{\sigma_x}(a-x)^r R_s(a)\,\mathrm{d}a=0$ by Cauchy theorem. □

Corollary A.5.

1. $R(a)=R(-a-1)$ for $a\notin E$ and $S(a)=S(-a-1)$ for $a\notin F$.
2. $\operatorname{Res}\limits_{a=\sqrt{k}+2t}R(a)=-\operatorname{Res}\limits_{a=-\sqrt{k}-2t-1}R(a)$ and $\operatorname{Res}\limits_{a=\sqrt{k}+2t+1}S(a)=-\operatorname{Res}\limits_{a=-\sqrt{k}-2t-2}S(a)$.
3. $\lim\limits_{\mathrm{Im}a\to\pm\infty}R(a)=1=\lim\limits_{\mathrm{Im}a\to\pm\infty}S(a)$ uniformly for $|Re\ a|\leqslant r$, in particular R and S are not identically zero.

Proof. We shall prove the statements about the R function. Similar proofs hold for the S function.

Note that $R_s(a)=R_s(-a-1)$ for all $a\notin E$ and $s\geqslant 1$, hence we get (1). Also $\operatorname{Res}\limits_{a=\sqrt{k}+2t}R_s(a)=-\operatorname{Res}\limits_{a=-\sqrt{k}-2t-1}R_s(a)$ for all $t\in\mathbf{Z}_+, s\geqslant 1$ and for all $x\in E$ we have $\operatorname{Res}\limits_{a=x}R(a)=\frac{1}{2\pi i}\int_{\sigma_x}R(z)\,\mathrm{d}z=\frac{1}{2\pi i}\int_{\sigma_x}R_s(z)\,\mathrm{d}z=\lim\limits_{s\to\infty}\operatorname{Res}\limits_{a=x}R_s(a)$, hence we get (2). Now suppose $|Re\ a|\leqslant r$, then $\frac{1}{|\theta_j-\lambda'|}\leqslant\frac{1}{(j-1)^2}$ for $j\geqslant j_0\geqslant r+1$ and $\frac{2t(2t-1)}{|\theta_t-\lambda'|}\leqslant\frac{4}{2|\mathrm{Im}a|}$ for $t\geqslant t_0\geqslant\sqrt{k}+r+2$, hence $\frac{2t(2t-1)}{|\theta_t-\lambda'|}\leqslant\frac{2t_0(2t_0-1)}{|\mathrm{Im}a|}=\frac{L}{|\mathrm{Im}a|}$ for all

$t, |\mathrm{Im}\ a| \geqslant 1$. Lemma A.3 gives

$$|R_s(a) - 1| \leqslant \prod_{j=2}^{s}\left(1 + \sum_{i=1}^{j-1}\left|\frac{\alpha_{2i}}{\theta_j - \lambda'} \prod_{t=j-i}^{j-1} \frac{2t(2t-1)}{\theta_t - \lambda'}\right|\right) - 1$$
$$\leqslant \exp \sum_{j=2}^{s}\sum_{i=1}^{j-1}\left|\frac{\alpha_{2i}}{\theta_j - \lambda'}\right|\left(\frac{L}{|\mathrm{Im}\, a|}\right)^i - 1$$
$$\leqslant \exp\left(\frac{j_0}{|\mathrm{Im}\, a|^2} + \frac{\pi^2}{6}\right)\left(|V|\left(\sqrt{\frac{L}{|\mathrm{Im}\, a|}}\right) - |V|(0)\right) - 1,$$

where $|V|(x) = \sum_{\iota \geqslant 0} |\alpha_{2i}| x^{2i}$ is an entire function, hence

$$|\mathrm{R}(a) - 1| \leqslant \exp\left(\frac{j_0}{|\mathrm{Im}\, a|^2} + \frac{\pi^2}{6}\right)\left(|\mathrm{V}|\left(\sqrt{\frac{L}{|\mathrm{Im} a|}}\right) - |\mathrm{V}|(0)\right) - 1 \to 0 \text{ as}$$

$|\mathrm{Im} a| \to \infty$ uniformly for $|Re\ a| \leqslant r$.

□

Proposition A.6. The polynomials $\left\{s^{1-\sqrt{k}} b_s(a) : s \geqslant 1\right\}$ $\left(\text{respectively, } \left\{s^{1-\sqrt{k}} c_s(a) : s \geqslant 1\right\}\right)$ converge uniformly on compact sets to a non-trivial entire transcendental function.

Proof. We shall prove the statement about the first set of polynomials. The statement about the second set of polynomials is proved similarly.

Note that by propositions A.1 and A.4 $\lim_{s\to\infty} s^{1-\sqrt{k}} b_s(a)$ exists for all $a \in \mathbf{C}$ and $\{s^{1-\sqrt{k}} b_s(a)\colon s \geqslant 1\}$ converge uniformly on compact sets $\subseteq \mathbf{C} - \{\sqrt{k} + 2t, -\sqrt{k} - 2t - 1 : t \in \mathbf{Z}_+\} = \mathbf{C} - E$, hence it suffices by corollary A.5 to prove that for all $x \in E$, $|\mathrm{s}^{1-\sqrt{\mathrm{k}}} b_{\mathrm{s}}(a)| \leqslant L_\rho < \infty$ for $|a - x| \leqslant 1/2\rho, 0 < \rho < 1/2, s \geqslant 1$.

We have $|s^{1-\sqrt{\mathrm{k}}} b_{\mathrm{s}}(a)| = \left|\frac{1}{2\pi \mathrm{i}} \int_{\sigma_{\mathrm{x}}} \frac{s^{1-\sqrt{\mathrm{k}}} b_s(z)}{z-a} \mathrm{d}z\right|$ for $|a - x| \leqslant 1/2\rho, s \geqslant 1$ where σ_x is the loop $\theta \to x + \rho \mathrm{e}^{i\theta}$, $0 < \rho < 1/2$ so that $|s^{1-\sqrt{\mathrm{k}}} b_s(a)| \leqslant L_\rho$ as desired if $|s^{1-\sqrt{\mathrm{k}}} b_{\mathrm{s}}(z)| \leqslant 1/2L_\rho$ for $|z - x| = \rho, s \geqslant 1$.

Next we turn to the determination of the zero sets of the entire functions $\lim_{s\to\infty} s^{1-\sqrt{k}} b_s(a)$ and $\lim_{s\to\infty} s^{1-\sqrt{k}} c_s(a)$. □

Proposition A.7.

$$\lim_{s\to\infty} s^{1-\sqrt{k}} b_s(a) = 0 \text{ iff } \lim_{s\to\infty} \min_{1\leqslant i\leqslant S} |a(a+1) - \sqrt{k} - k - \beta_{\mathrm{i}}^{(\mathrm{s})}| = 0$$
$$\text{iff } \liminf_{s\to\infty} \min_{1\leqslant i\leqslant S} |a(a+1) - \sqrt{k} - k - \beta_{\mathrm{i}}^{(s)}| = 0 \text{ and}$$
$$\lim_{s\to\infty} s^{1-\sqrt{k}} c_s(a) = 0 \text{ iff } \lim_{s\to\infty} \min_{1\leqslant i\leqslant S} |a(a+1) - \sqrt{k} - k - \gamma_{\mathrm{i}}^{(s)}| = 0$$
$$\text{iff } \liminf_{s\to\infty} \min_{1\leqslant i\leqslant s} |a(a+1) - \sqrt{k} - k - \gamma_{\mathrm{i}}^{(s)}| = 0.$$

Proof. We shall prove the first statement and the second one is proved similarly.

Let $T(a_1) = \lim_{s\to\infty} s^{1-\sqrt{k}} b_s(a_1) \neq 0$ for $0 < |a_1 - a| \leqslant r_0$, then by proposition A.6, for all $0 < r \leqslant r_0$, $s^{1-\sqrt{k}} b_s(z)$ has no zeros on $\partial \overline{B}_r(a)$ for $s \geqslant s_0(r)$ and $\frac{1}{2\pi i}\int_{\partial \overline{\mathrm{B}}_{\mathrm{r}}(a)} \frac{T'(z)}{T(z)} \mathrm{d}z = \lim_{s_0(r)\leqslant s\to\infty} \frac{1}{2\pi i}\int_{\partial \overline{\mathrm{B}}_{\mathrm{r}}(a)} \frac{(s^{1-\sqrt{k}} b_s(z))'}{s^{1-\sqrt{k}} b_s(z)} \mathrm{d}z$. This together with the fact that the map $a \to a(a+1) - \sqrt{k} - k$ is an open map shows that the zeros of the T function are determined by the stated conditions. □

Now we have the following theorem.

Theorem A.8.

1. $\lim_{s\to\infty} s^{1-\sqrt{k}} b_s(\lambda')$ exists for all $\lambda' \in \mathbf{C}$ and

$$\lim_{s\to\infty} s^{1-\sqrt{k}} b_s(\lambda') = 0 \text{ iff } \lim_{s\to\infty} \min_{1\leqslant i\leqslant s} |\lambda' - \beta_{\mathrm{i}}^{(s)}| = 0$$
$$\text{iff } \liminf_{s\to\infty} \min_{1\leqslant i\leqslant s} |\lambda' - \beta_{\mathrm{i}}^{(s)}| = 0.$$

2. $\lim_{s\to\infty} s^{1-\sqrt{k}} c_s(\lambda')$ exists for all $\lambda' \in \mathbf{C}$ and

$$\lim_{s\to\infty} s^{1-\sqrt{k}} c_s(\lambda') = 0 \text{ iff } \lim_{s\to\infty} \min_{1\leqslant i\leqslant s} |\lambda' - \gamma_{\mathrm{i}}^{(s)}| = 0$$
$$\text{iff } \liminf_{s\to\infty} \min_{1\leqslant i\leqslant s} |\lambda' - \gamma_{\mathrm{i}}^{(s)}| = 0.$$

□

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